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Eigenvector Scaling for Mode Localization in Vibrating Systems

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A new closed-loop mode localization technique known as eigenvector scaling is presented. Eigenvector scaling is a form of eigenstructure placement that uses feedback to alter portions of the original system eigenvectors. When properly applied, this technique restricts energy propagation in a vibrating system. Active eigenvector scaling also permits the development of an analytic solution for the closed-loop mode-localized discrete system. The ability to produce analytic solutions for both the controlled and the uncontrolled systems permits a direct comparison of the absolute displacements between the two cases. It is demonstrated that closed-loop mode localization can be used to restrict vibrations in certain regions of a discrete structure. Three examples are provided that show how this feedback control technique may be applied to multi-degree-of-freedom simple spring-mass and spring-mass-damper systems, as well as to lumped models of more complicated flexible structures.

Introduction

VIBRATION of flexible structures is a common problem in dynamics and controls. The relative ease with which vibrational energy may propagate through complex systems can lead to excessive displacements at critical locations, which in turn can lead to reduced performance or failure. It is not always necessary to control vibrations throughout an entire structure, but rather controlling certain areas within the structure that contain sensitive instruments or machines may be most critical. The motivation for this work, then, is to seek vibration control methods that produce quiet areas within a vibrating system subjected to arbitrary disturbances.

One way of controlling flexible structure vibration is through mode localization. Mode localization restricts motion by containing vibrational energy within a small area of the total system. This phenomenon was originally described in solid state physics[1] but also exists in flexible structures.[2,3] Mode-localized structures exhibit areas of comparatively high displacements, with amplitudes in these parts potentially several magnitudes greater than in the rest of the system. In a large number of studies, mode localization has been considered to be unfavorable, as in mistuned rotor blades.[4] However, some studies maintain that mode localization may be useful, since it confines system motion within a specific area, preventing the transmission of potentially harmful vibrational energy.[5] Additionally, if vibrational energy is contained, then it may be more easily controlled with damping or cancellation.

Loosely coupled, slightly mistuned flexible structures are especially susceptible to mode localization.[2,6,7] As a result, studies have concentrated on the passive mistuning of this type of structure. It is, however, also possible to create mode localization with active, closed-loop feedback control using a technique known as eigenstructure placement, which is the approach used in this paper.

Eigenstructure placement of multi-input/multi-output systems originated with pole (eigenvalue) assignment in controlled systems.[8,9] Further work demonstrated that a system is not completely described by eigenvalues and that some freedom in choosing system eigenvectors is permitted.[10–13] More recently, this technique has been successfully used to generate mode localization in a lumped-mass system.[14] This was done by placing the closed-loop mode shapes so that all eigenvectors exhibited a proportionally high degree of vibration at a single location in the system. This paper continues and expands that work.

The objectives of this paper are twofold. The first objective is to introduce a mode localization technique called eigenvector scaling. Eigenvector scaling is a form of eigenstructure placement that adjusts specific portions of each eigenvector, uniformly increasing the relative displacement of selected areas of the vibrating system. The second objective of this paper is to focus on the time domain response of a vibrating system to study the effects of eigenvector scaling. Mode localization studies have focused primarily on the placement of the eigenstructure and on the relative amplitudes described by the eigenvectors. Little attention has been given to the system's actual amplitudes of vibration, so it is not possible to tell from the eigenvectors whether absolute system displacements have increased, decreased, or remained unchanged. As this paper will show, eigenvector scaling successfully produces mode localization and, given the right conditions, can produce vibration isolation, or areas of reduced vibration, frequently with a minor amount of control effort.

This analysis is divided into two parts. First, the control system is developed, with the localization procedure outlined and discussed, including the behavior of the system and its analytic solution. Second, three examples are provided that support the results found in the control system development.

Control System

The controlled system model development is divided into three sections. The first section reviews general eigenstructure placement procedures using a discrete system model. A lumped-mass approach is used because it permits a straightforward examination of the proposed control approach, which may then be applied to both simple spring-mass structures as well as lumped-parameter models of continuous-parameter systems. The second section introduces eigenvector scaling and discusses its effect on the control gains and the system description. The third section develops the analytic solution for the closed-loop mode-localized system and compares this result to the open-loop analytic solution.

Eigenstructure Placement

Consider the n-degree-of-freedom (DOF) lumped-mass system given in Fig. 1, with equations of motion,

$$[M]\{\ddot{y}\} + [C_{\text{damp}}]\{\dot{y}\} + [K_{\text{spring}}]\{y\} = \{\xi\} \tag{1}$$

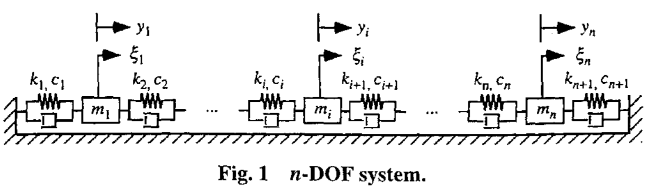

Fig. 1 n-DOF system.

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where $[M]$ is the diagonal mass matrix, $[C_{\text{damp}}]$ is the damping coefficient matrix, and $[K_{\text{spring}}]$ is the stiffness matrix. The vectors $\{\ddot{y}\}$, $\{\dot{y}\}$, and $\{y\}$, respectively, represent the acceleration, velocity, and displacement coordinates. The vector $\{\xi\}$ represents the external disturbance inputs. Equation (1) may be rewritten as

$$[I]\{\ddot{y}\} + [C_m]\{\dot{y}\} + [L]\{y\} = [M]^{-1}\{\xi\} = \{F\} \tag{2}$$

The preceding system equations of motion may also be written in state-space form, given $\{x\}^T = [\{y\}^T\{\dot{y}\}^T]$:

$$\{\dot{x}\} = [A]\{x\} + [B]\{u\} \qquad \{y\} = [C]\{x\} \tag{3}$$

with

$$[A] = \begin{bmatrix} 0 & I \\ -L & -C_m \end{bmatrix} \qquad [B] = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tag{4}$$

$$[C] = [I \quad 0] \tag{5}$$

This form of the system equations of motion has eigenstructure such that

$$[A][\nu] = [\nu][\omega] \tag{6}$$

where $[\omega]$ is a diagonal matrix containing the eigenvalues and $[\nu]$ contains eigenvectors $\{\phi\}$ in the form

$$\nu_{2i-1} = \begin{Bmatrix} \phi_i \\ \omega_{2i-1}\phi_i \end{Bmatrix} \qquad \nu_{2i} = \begin{Bmatrix} \phi_i \\ \omega_{2i}\phi_i \end{Bmatrix} \tag{7}$$

Eigenstructure placement of the system defined by Eqs. (3) is performed by altering the eigenvalues and eigenvectors given by Eq. (6). This may be accomplished by developing a closed-loop control system. If control inputs are functions of state variables $\{u\} = [K]\{x\}$, they may be added to the uncontrolled external disturbance forces $\{\xi\}$ such that the total input forces become $[K]\{x\}+\{\xi\}$. Equation (4) may then be rewritten as

$$\{\dot{x}\} = [A]\{x\} + [B_K][K]\{x\} + [B_\xi]\{\xi\} \tag{8}$$

or be rearranged as

$$\{\dot{x}\} = [A_c]\{x\} + [B_\xi]\{\xi\} \tag{9}$$

with the closed-loop description $[A_c] = [A] + [B_K][K]$. Note that although $[B_K]$ and $[B_\xi]$ are both of the general form given by Eq. (4), they are kept separate to emphasize that the control forces need not necessarily be applied at the same location as the disturbance inputs. Further, if the controlled system eigenvalues are contained in the left half complex plain (stable) and the eigenvectors are linearly independent and of the form given by Eq. (7), then by Eq. (6),

$$[A_c] = [\nu_c][\omega_c][\nu_c]^{-1} \tag{10}$$

It is assumed for simplicity that all eigenvectors and eigenvalues are controllable. The necessary gains required to achieve this system may be calculated by

$$[K] = [B_{L^+}]^{-1}([A_c] - [A]) \tag{11}$$

with $[B_{L^+}]^{-1} = ([B_K]^T[B_K])^{-1}[B_K]^T$.

Eigenvector Scaling

Next, consider a special case of eigenstructure placement, where the closed-loop solution has the following characteristics:

$$[\omega_c] = [\omega] \qquad [\nu_c] = [\Delta][\nu] \tag{12}$$

The matrix $[\Delta]$ is a $2n \times 2n$ diagonal matrix composed of scaling elements d_i, $i = 1$ to n, where, because $[\nu]$ is of the form given by Eq. (7), $d_{n+i} = d_i$. In general, the scaling factors are real numbers such that $d_i \neq 0$. However, the scaling factors used in this paper are restricted to $d_i \geq 1$ to simplify the analysis. Thus, from Eq. (12) the closed-loop eigenvalues are the same as the open-loop eigenvalues, whereas the closed-loop eigenvectors are the open-loop eigenvectors scaled at elements i and $n+i$ by d_i.

To further simplify, let $d_i = d_{n+i} = 1$ for all $i \neq k$ and $d_i = d_{n+i} = d$ for all $i = k$, where k is some arbitrary whole number such that $1 \leq k \leq n$. The eigenvector scaling method does not in general require this restriction; this form of $[\Delta]$ is used because it provides a convenient reference with which to clarify the subsequent equations. By substituting Eq. (12) into Eq. (10) one can see that the controlled system matrix $[A_c]$ becomes

$$[A_c] = [\Delta][\nu][\omega][\nu]^{-1}[\Delta]^{-1} = [\Delta][A][\Delta]^{-1} \tag{13}$$

Using the proposed form of $[\Delta]$ in Eq. (13) produces an interesting result. Since columns k and $n+k$ are divided by d and rows k and $n+k$ are multiplied by d, the only elements of the original matrix $[A]$ that are affected by this operation are the coupling terms in columns and rows k and $n+k$, as illustrated in Fig. 2. This result can be directly related to the stiffness and damping matrices given by Eq. (4a):

$$[L_c] = [D][L][D]^{-1} \qquad [C_{mc}] = [D][C_m][D]^{-1} \tag{14}$$

where $[D]$ is an $n \times n$ matrix of scaling elements d_i, $i = 1$ to n such that

$$[\Delta] = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \tag{15}$$

The control system affects the stiffness and damping coupling terms at the interface between the isolated, unscaled sections, where $d = 1$, and localized or scaled portions of the system, where $d \geq 1$. For the isolated side, these coupling terms have been reduced or softened by a factor d, whereas on the localized side, these same terms have been magnified or stiffened by d. Further, if the stiffness

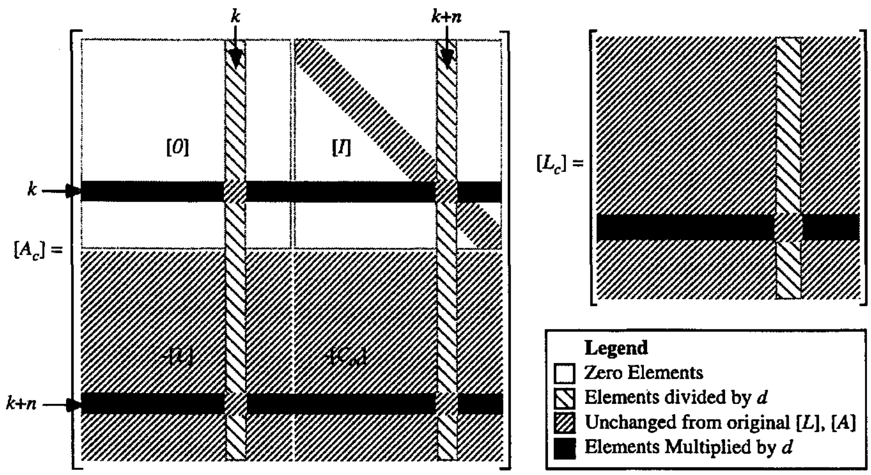

Fig. 2 Eigenvector-scaled system description matrices obtained from first-order ([A]) and second-order ([L]) equations of motion. Eigenvectors scaled by d at elements k and $k+n$ affect very few terms in the system description matrix.

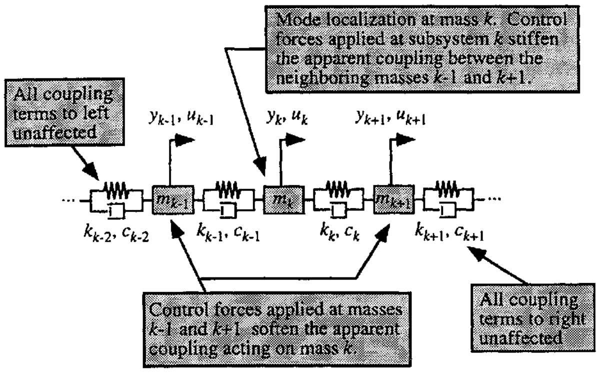

Fig. 3 Effect of eigenvector scaling on mass displacement coupling. Mass k is localized, so that only coupling (stiffness and damping) terms $k-1$ and k are affected.

and damping matrices are tridiagonal, that is, if the coupling between each mass is limited to the action of neighboring masses, then at most eight elements in matrix $[A]$ of Eq. (13) are altered, with only four elements in each of the matrices given in Eq. (14) affected. Also, it can be seen from Eq. (11), from Figs. 2 and 3, and intuitively from Eqs. (14) that the gains required to achieve this new set of stiffness and damping terms are, respectively, functions of displacement and velocity. Physically, this sparsely populated gain matrix indicates that if the stiffness and damping matrices are tridiagonal, then control actuators need only be placed on either side of the localization interface, as illustrated in Fig. 3.

Analytic Solution

The analytic solution of a second-order set of differential equations of motion is presented here. This solution provides actual system displacements, which permits an absolute analysis of the system response, since eigenvectors themselves only indicate relative vibration amplitudes. To determine the analytic solution, we will consider a system with no damping and we will use the second-order form of the differential equations of motion. This approach is chosen because there exists an analytic solution for this form that is relatively easily analyzed and that may be physically interpreted.

If there are no damping terms, then $[C_m]=0$, and the second-order equations given by Eq. (2) may be reduced to

$$[I]\{\ddot{y}\}+[L]\{y\}=\{F\} \tag{16}$$

This system is described by the eigenstructure

$$[L][\phi]=[\phi][\lambda] \tag{17}$$

In the state space, the absence of damping produces a simplified form of Eq. (4):

$$[A]=\begin{bmatrix}0 & I\\ -L & 0\end{bmatrix} \tag{18}$$

The eigenstructure of Eq. (18) is given by Eqs. (6) and (7), and the eigenvalues of the first- and second-order forms are related by

$$\lambda_i=\omega_{2i-1}\omega_{2i}=\bar{\omega}_{2i}\omega_{2i}$$

As with Eq. (10), the second-order closed-loop system description defined by

$$[L_c]=[\phi_c][\lambda_c][\phi_c]^{-1} \tag{19}$$

may be achieved by applying the control forces $[K_2]\{y\}$ such that the gains $[K_2]$ are defined by

$$[K_2]=[M]([L]-[L_c]) \tag{20}$$

If no damping is introduced, then the control forces are functions of displacement only, with coefficients that are related to the original system stiffnesses. Further, if eigenvector scaling is used, then the gains are related to the system coupling stiffnesses only and are scaled as described in the last paragraph of the section discussing eigenvector scaling.

The equations of motion for the n-DOF system given by Eq. (16) may be rewritten using principal coordinates $\{q\}$ and the relationship

$$\{y\}=[\Phi]\{q\} \tag{21}$$

where $[\Phi]$ are the mass-normalized system eigenvectors giving

$$[\Phi]^T[M][\Phi]=[I] \qquad [\Phi]^T[K][\Phi]=[\lambda] \tag{22}$$

Using the relationships given in Eqs. (21) and (22), one may write Eq. (16) in terms of principal coordinates, with each equation solved independently. Further, by assuming that the disturbances are of the form $f_i\sin(\omega_{fi}t+\varphi_i)$, these solutions maintain a tight closed form, which may be transformed back into the original coordinates using Eqs. (21) and (22):

$$\begin{aligned}\{y(t)\}&=[\Phi][Cs][\Phi]^T[M]\{y_0\}+[\Phi][Sn][\lambda]^{-\frac{1}{2}}[\Phi]^T[M]\{\dot{y}_0\}\\&\quad-[\Phi][Cs]([\Omega]\cdot[\Phi]^T)[\sin(\varphi)]\{f\}-[\Phi][Sn][\lambda]^{-\frac{1}{2}}\\&\quad\times([\Omega]\cdot[\Phi]^T)[\cos(\varphi)]\{f\}+[\Phi]([\Omega]\cdot[\Phi]^T)\{\xi\}\end{aligned} \tag{23}$$

with diagonal matrices $[Cs]$ and $[Sn]$, respectively, composed of elements $\cos(\lambda_i^{1/2}t)$ and $\sin(\lambda_i^{1/2}t)$. The matrix $[\Omega]$ is composed of elements $\Omega_{ij}=(\lambda_i-\omega_{fj}^2)^{-1}$, with the operation $[\Omega]\cdot[\Phi]^T$ returning an $n\times n$ matrix containing elements $\Omega_{ij}\Phi_{ij}^T$, $i, j=1$ to n, with no sum on i or j. The diagonal matrices $[\sin(\varphi)]$ and $[\cos(\varphi)]$ are, respectively, composed of the sine and cosine of the disturbance phase angles φ_i; $[\omega_f]$ is a diagonal matrix composed of the disturbance frequencies ω_{fi}; and $\{f\}$ is a vector of the disturbance input coefficients f_i.

To facilitate comparison between the uncontrolled and controlled systems, one can write the preceding equation in simpler form by grouping terms:

$$\{y(t)\}=[Y_y]\{y_0\}+[Y_{\dot{y}}]\{\dot{y}_0\}-[Y_{yf}]\{f\}-[Y_{\dot{y}f}]\{f\}+[Y_\xi]\{\xi\} \tag{24}$$

where the matrices $[Y]$ contain information about the absolute system displacements and how they are affected by the external inputs. By comparing Eqs. (23) and (24), it can be seen that the first two sets of terms on the right-hand side of Eq. (24) denote the system response caused by initial displacement and velocity conditions. The next two terms reflect the system response caused by the initial effects of the disturbance inputs $\{\xi\}$. These first four terms make up the system transient solution. The fifth term in Eq. (24) denotes the system response to forced inputs and is the steady-state portion of the analytic solution.

For the closed-loop, eigenvector-scaled system, the equations of motion for a spring-mass system given by Eq. (16) may be written as

$$[I]\{\ddot{y}\}+[L_c]\{y\}=\{F\} \tag{25}$$

Once the eigenvectors of a system are altered, the analytic solution given by Eq. (25) may no longer be used, since Eqs. (22) are no longer valid. However, with eigenvector scaling it is possible to write Eq. (25) as

$$[M_c]\{\ddot{y}\}+[K_c]\{y\}=[M_c]\{F\} \tag{26}$$

such that

$$[M_c]=[D]^{-1}[M][D]^{-1} \qquad [K_c]=[D]^{-1}[K][D]^{-1} \tag{27}$$

It is important to realize that Eqs. (27) do not reflect true changes in system mass and stiffness but rather are used to write the equations of motion in the appropriate form. That is, from Eq. (27) it can be seen that

$$[M_c]^{-1}[K_c]=[D][L][D]^{-1}=[L_c] \tag{28}$$

and, since $[\Phi_c] = [D][\Phi]$, it can be seen from Eqs. (22) and (27) that

$$[\Phi_c]^T[M_c][\Phi_c] = [I] \qquad [\Phi_c]^T[K_c][\Phi_c] = [\lambda] \tag{29}$$

which is equivalent to the conditions given by Eqs. (22). Thus, Eq. (26) may be used to produce the equation for the controlled system displacements by using the techniques for the uncontrolled system outlined earlier:

$$\begin{aligned}\{y(t)\} &= [D][Y_y][D]^{-1}\{y_0\} + [D][Y_{\dot{y}}][D]^{-1}\{\dot{y}_0\} \\ &\quad - [D][Y_{yf}][D]^{-1}\{f\} - [D][Y_{\dot{y}f}][D]^{-1}\{f\} \\ &\quad + [D][Y_\xi][D]^{-1}\{\xi\}\end{aligned} \tag{30}$$

By comparing Eq. (30) to Eq. (24), one can see that the analytic solution for the closed-loop system is the same as the solution for the open-loop system, except that Eq. (30) contains scaling matrices $[D]$ and $[D]^{-1}$. For the uncontrolled system, the scaling matrices $[D]$ and $[D]^{-1}$ become the identity matrix, with all scaling terms set to $d_i = 1$. For the controlled, mode-localized system, these scaling matrices multiply or divide certain components of the displacement coefficients by d. With localization occurring at k, matrix $[D]$ multiplies coefficients in row k of $[Y]$ by d. Additionally, the matrix $[D]^{-1}$ divides column k of $[Y]$ by d. Physically, the scaling term $[D]$ implies that the absolute amplitude of mass k has increased by a factor of d, except for the contribution of terms containing y_{0k}, $\dot{y}_{0k}$, or f_k, which remain unchanged because of the presence of $[D]^{-1}$. This means that the effect of any disturbance introduced outside the area of localization will be magnified d times within the area of localization, with the control system acting as an amplifier. In turn, the scaling matrix $[D]^{-1}$ and its operation on the terms containing y_{0k}, $\dot{y}_{0k}$, or f_k reduce the contribution of the input conditions introduced at the localized masses k. Thus, the effect of any disturbance introduced inside the area of localization will be reduced by d outside the area of localization, and the control system acts as an isolator.

Control Effort Required to Produce Eigenvector Scaling

In the most general case, the control force needed to apply eigenvector scaling is given by

$$\{u\} = [K]\{x_c\} = [B_{L+}]^{-1}([\Delta][A][\Delta] - [A])\{x_c\} \tag{31}$$

where $\{x_c\}$ are the controlled system displacements. Because of the nature of eigenvector scaling, $\{x_c\}$ may be related to the open-loop states by

$$\{x_c\} = [\Delta]\{x\}\kappa \tag{32}$$

for an arbitrary disturbance at a specified location. Here $[\Delta]$ defines the relative displacement between the uncontrolled and localized states, whereas κ gives the appropriate amplitude, which is dependent on the location of the disturbance. For example, if the disturbance is contained in the area of localization, then $\kappa = 1/d$, for reduced displacements in the isolated region. If the disturbance is contained in the area of isolation, then $\kappa = 1$, for increased amplitudes in the localized region. Equation (31) may then be expressed as

$$\{u\} = [B_{L+}]^{-1}([\Delta][A][\Delta] - [A])[\Delta]\{x\}\kappa \tag{33}$$

so it can be seen that the necessary control force $\{u\}$ generally is dependent upon the system parameters, the scaling matrix $[\Delta]$, the disturbance location, and the magnitude of the original system states.

To simplify the preceding expression, take a spring-mass system that may be described using a diagonal mass matrix (as assumed in this paper). Equation (33) becomes

$$\{u\} = ([K] - [D][K][D]^{-1})[D]\{y\}\kappa \tag{34}$$

where $\{u\}$ is now dependent only on the coupling stiffness separating the isolated region from the localized region, system displacements $\{y\}$, scaling matrix $[D]$, and the location of the disturbance, which dictates κ.

Examples

Three examples are provided here to help support the preceding derivations. Example 1 demonstrates how the analytic solution changes with eigenvector scaling. Example 2 shows that eigenvector scaling may be applied to a higher-DOF tridiagonal spring-mass-damper system to produce mode localization using only two control actuators. Example 3 shows that eigenvector scaling may be applied to lumped-mass models representing realistic systems with more complicated coupling.

Example 1

Take as an example the two-DOF system given by Fig. 4a. With masses $m = 2$ kg, stiffnesses $k = 50$ N/m, this system may be described by

$$[L] = [M]^{-1}[K] = \begin{bmatrix} 50 & -25 \\ -25 & 50 \end{bmatrix} \tag{35}$$

with eigenstructure

$$[\lambda] = \begin{bmatrix} 8.6^2 & 0 \\ 0 & 5^2 \end{bmatrix} \qquad [\phi] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{36}$$

By assuming initial displacement conditions y_{0i} and applied forcing functions $f_i = \sin(6t)$ N, one can expand Eq. (23) to give the system displacements, in meters,

$$\begin{aligned}\{y(t)\} &= \frac{1}{2}\begin{bmatrix} cs_1 + cs_2 & -cs_1 + cs_2 \\ -cs_1 + cs_2 & cs_1 + cs_2 \end{bmatrix}\begin{Bmatrix} y_{01} \\ y_{02} \end{Bmatrix} \\ &\quad - \frac{1}{2}\begin{bmatrix} 0.1sn_1 - 0.3sn_2 & -0.1sn_1 - 0.3sn_2 \\ -0.1sn_1 - 0.3sn_2 & 0.1sn_1 - 0.3sn_2 \end{bmatrix}\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \\ &\quad - \begin{bmatrix} 0.02 & 0.03 \\ 0.03 & 0.02 \end{bmatrix}\begin{Bmatrix} f_1 \sin(6t) \\ f_2 \sin(6t) \end{Bmatrix}\end{aligned} \tag{37}$$

where $cs_1 = \cos(8.6t)$, $cs_2 = \cos(5.0t)$, $sn_1 = \sin(8.6t)$, and $sn_2 = \sin(5.0t)$.

For localization $d = 10$ at mass 2, the controlled system has an eigenstructure as described in Eq. (12):

$$[\lambda_c] = \begin{bmatrix} 8.6^2 & 0 \\ 0 & 5^2 \end{bmatrix} \qquad [\phi_c] = \begin{bmatrix} 1 & 1 \\ -10 & 10 \end{bmatrix} \tag{38}$$

Note that the controlled eigenvalues are the same as in the uncontrolled system, whereas the relative displacement of mass 2 has

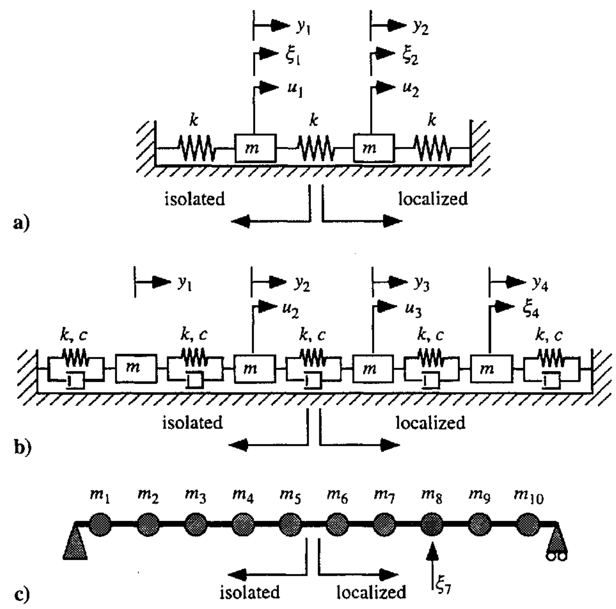

Fig. 4 a) Two-DOF spring-mass system used in example 1, b) 4-DOF spring-mass-damper system used in example 2, and c) 10-DOF spring-mass model of a simply supported beam used in example 3.

increased by $d = 10$, as shown by the controlled eigenvectors. The new controlled system expression is obtained using Eq. (14):

$$[L_c] = \begin{bmatrix} 50 & \mathbf{-2.5} \\ \mathbf{-250} & 50 \end{bmatrix} \tag{39}$$

The scaled elements in the preceding equation are shown in boldface type, with the required gains

$$[K_2]\{y\} = \begin{bmatrix} 0 & -45 \\ 450 & 0 \end{bmatrix}\{y\} \tag{40}$$

found by using Eq. (20). Note that the coupling term in the first row of Eq. (39) (element 1, 2) is one-tenth of the original value shown in Eq. (35), so that the coupling forces felt by the first mass are 10 times less than in the uncontrolled case. Conversely, the coupling term in the second row of Eq. (39) (element 2, 1) is 10 times stiffer than in the uncontrolled system, and so the controlled coupling forces felt by the second mass are now 10 times greater than in the uncontrolled case.

By Eq. (30), the controlled system has the following absolute displacements:

$$\{y(t)\} = \frac{1}{2}\begin{bmatrix} cs_1 + cs_2 & \mathbf{-0.1}cs_1 + \mathbf{0.1}cs_2 \\ \mathbf{-10}cs_1 + \mathbf{10}cs_2 & cs_1 + cs_2 \end{bmatrix}\begin{Bmatrix} y_{01} \\ y_{02} \end{Bmatrix}$$
$$- \frac{1}{2}\begin{bmatrix} 0.1sn_1 - 0.3sn_2 & \mathbf{-0.01}sn_1 - \mathbf{0.03}sn_2 \\ \mathbf{-1.0}sn_1 \mathbf{-3.0}sn_2 & 0.1sn_1 - 0.3sn_2 \end{bmatrix}\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$
$$- \begin{bmatrix} 0.02 & \mathbf{0.003} \\ \mathbf{0.30} & 0.02 \end{bmatrix}\begin{Bmatrix} f_1 \sin(6t) \\ f_2 \sin(6t) \end{Bmatrix} \tag{41}$$

with the elements that are different from those of the uncontrolled system response shown in boldface type. Compared with the uncontrolled displacements given in Eq. (37), element 1, 2 of each controlled displacement response matrix has been divided by $d = 10$. This means that any system excitation originating within the area of localization (at mass 2) will be reduced outside this area, effectively isolating mass 1. Also, element 2, 1 of each displacement response matrix has been multiplied by 10, which means that a disturbance applied outside the area of localization will produce a tenfold increase in the amplitude within the area of localization.

For the preceding example with disturbances on mass 2, Eq. (34) becomes

$$\{u\} = \begin{bmatrix} 0 & (d-1)k/d \\ (1-d)k & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}\{y\}\frac{1}{d}$$
$$= \begin{bmatrix} 0 & (1-1/d)k \\ -(1-1/d)k & 0 \end{bmatrix}\{y\} \tag{42}$$

In this case, $\{u\}$ depends only on the degree of localization d, the magnitude of the coupling stiffness k, and the uncontrolled system displacement levels of y_1 and y_2. Thus, stiff coupling, a high degree of localization, and relatively large uncontrolled system displacements will require more control effort than if the coupling is soft, the degree of localization is low, and the uncontrolled system displacements are small. It can also be seen from Eq. (42) that the actual control effort will always be less than $k\{y\}$. Incidentally, Eq. (42) applies to any n-DOF tridiagonal spring-mass system if $[D]$ is a simple step and $\{y\}$ includes the displacements on the interface between the isolated and localized sections of the system.

Example 2

Next, consider the four-DOF system given in Fig. 4b, with masses $m = 1$, damping factors $c = 0.4$, and stiffnesses $k = 2$, so that the system description matrix may be expressed in state-space form shown in Eq. (4) with

$$[-L \quad -C_m] =$$
$$\begin{bmatrix} -4 & 2 & 0 & 0 & -0.8 & 0.4 & 0 & 0 \\ 2 & -4 & 2 & 0 & 0.4 & -0.8 & 0.4 & 0 \\ 0 & 2 & -4 & 2 & 0 & 0.4 & -0.8 & 0.4 \\ 0 & 0 & 2 & -4 & 0 & 0 & 0.4 & 0.8 \end{bmatrix} \tag{43}$$

Scale the relative displacements of masses 3 and 4 by a factor $d = 2$ using Eq. (13) so that the stiffness and damping terms of the controlled system description $[A_c]$ become

$$[-L_c \quad -C_{mc}] =$$
$$\begin{bmatrix} -4 & 2 & 0 & 0 & -0.8 & 0.4 & 0 & 0 \\ 2 & -4 & 0.7 & 0 & 0.4 & -0.8 & 0.14 & 0 \\ 0 & 6 & -4 & 2 & 0 & 1.2 & -0.8 & 0.4 \\ 0 & 0 & 2 & -4 & 0 & 0 & 0.4 & -0.8 \end{bmatrix} \tag{44}$$

Notice that only four elements of the system description matrix have been altered, so that the required gain matrix

$$[K] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.3 & 0 & 0 & 0 & -0.26 & 0 \\ 0 & 4.0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{45}$$

is sparsely populated. Control actuation occurs exclusively at masses 2 and 3, affecting only the coupling between these two subsystems. With just two actuators, this control is sufficient to reduce or to isolate the response of both masses 1 and 2 caused by any external disturbances exerted on either mass 3 or 4. In fact, this two-control-actuator containment effect holds for any tridiagonal system, regardless of the actual number of DOF.

This containment is demonstrated in Fig. 5. The left-hand plots in Fig. 5 show the response of the uncontrolled and mode-localized systems to a disturbance on mass 4 of $\xi_4 = \sin(2t)$. It can be seen that, although the relative shape of the response of each system remains unchanged, the absolute amplitudes of the masses 1 and 2 in the controlled system have been affected by localization. For the mode-localized system, the absolute amplitudes of the isolated region (masses 1 and 2) have been reduced by a factor $d = 2$, whereas the displacements of the localized region (masses 3 and 4) remain unchanged from the uncontrolled case. The bottom plot shows the required control forces to produce the localized response. Note that these forces are considerably less than the applied force.

To show the effects of parameter variation on the control results, this example was repeated where the controller was applied to a system whose parameters were significantly altered from the original system. This is not intended to be a comprehensive study of parameter sensitivity, but it does demonstrate for this example how the controller responds to modeling errors.

The four-DOF spring-mass-damper system used earlier was changed so that the coupling stiffness between the localized and isolated portions of the structure (the stiffness between masses 2

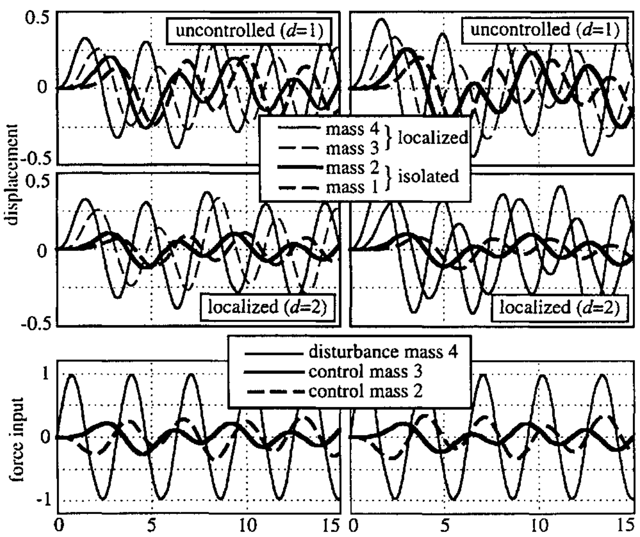

Fig. 5 Four-DOF spring-mass-damper system used in example 2 subjected to a $1/\pi$ Hz disturbance on element 4, with elements 1 and 2 (shown with boldface lines) isolated. Model includes both basic formulation (left) and the addition of parameter uncertainty (right).

Table 1 Eigenvalues of system used in example 2, showing effect of parameter uncertainty on uncontrolled system and localized (d = 2) system, the system remained stable, even with 15% coupling uncertainty between the localized and isolated regions

Eigenvalue	Basic model uncontrolled/localized	Uncontrolled with parameter variation	Localized ($d = 2$) with parameter uncertainty
First	$-0.077 \pm 0.871i$	$-0.053 \pm 0.748i$	$-0.060 \pm 0.8791i$
Second	$-0.276 \pm 1.639i$	$-0.202 \pm 1.356i$	$-0.200 \pm 1.349i$
Third	$-0.524 \pm 2.228i$	$-0.363 \pm 1.872i$	$-0.366 \pm 1.881i$
Fourth	$-0.724 \pm 2.591i$	$-0.553 \pm 2.310i$	$-0.545 \pm 2.294i$

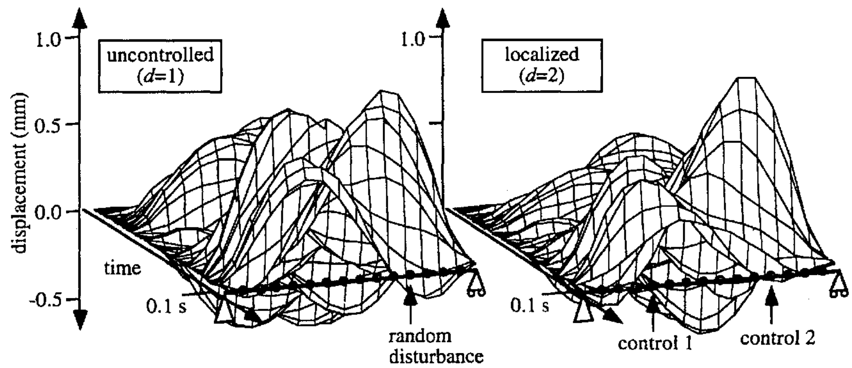

Fig. 6 Active mode localization applied to the simply supported beam used in example 3, with the left half of the beam isolated by a factor d = 2. Feedback gain is generated using a 2-DOF model and applied to the 14-DOF system shown.

and 3) was changed by 15%. In addition, the remaining system parameters were altered by as much as 35%, as shown in Eq. (46),

$$[-L \quad -C_m] = \begin{bmatrix} -4.0 & 1.8 & 0 & 0 & -0.9 & 0.4 & 0 & 0 \\ 1.8 & -4.7 & 1.7 & 0 & 0.4 & -0.9 & 0.3 & 0 \\ 0 & 1.7 & -4.4 & 1.9 & 0 & 0.3 & -0.9 & 0.4 \\ 0 & 0 & 1.9 & -4.1 & 0 & 0 & 0.4 & -0.7 \end{bmatrix} \tag{46}$$

which can be directly compared with Eq. (43). The eigenvalues for the perturbed and the original system are shown in Table 1. Note that parameter variation shifts system eigenvalues by as much as 18%. However, it can also be seen that applying the control law that was designed for the original system to the model with parameter variation still produces a stable system. In addition, the right-hand plots of Fig. 5 show that the perturbed system is successfully mode-localized in spite of the parameter variation. Note that the plots look similar to those of the original system, although the frequencies are somewhat different. The important thing to note is that the active localization controller is successful in reducing the displacements of masses 1 and 2, even with the introduction of parameter uncertainty.

Example 3

Finally, consider the model of a simply supported steel beam given in Fig. 4c, with dimensions $3 \times 25 \times 610$ mm ($\frac{1}{8} \times 1 \times 24$ in.). This simply supported beam is represented by a 14-DOF spring-mass model, which is obtained by dividing the beam into 14 equal sections (as shown in Fig. 4c) and inverting the resulting flexibility matrix.

As in the previous examples, a simple step scaling function is applied to the system description given in Eq. (28) to generate the controller. In this example, however, a 2-DOF model of the beam is used to generate the control, which is applied to the 14-DOF model described earlier. The 2-DOF model has masses at locations of elements 4 and 11 from Fig. 4c. To form the control law from Eq. (28), the relative displacements of the mass in the localized area are multiplied by 2, effectively isolating the other mass.

Numerical simulations of the system response to a random disturbance for both the uncontrolled and the eigenvector-scaled systems are given in Fig. 6. Note that even with a random disturbance, the absolute displacements in the isolated region (masses 1–7) have been reduced by a factor of 2, whereas the displacements in the localized area (masses 8–14) remain unchanged with the control application.

The required gain set for this case is

$$[K] = \begin{bmatrix} 0 & -0.5 \\ 1.0 & 0 \end{bmatrix} 742EI \tag{47}$$

For the excitation case given in Fig. 6, this gain set generates control forces of 6 N or less, which is on the order of the ±5 N random disturbance. Note that the preceding gain set is 2×2 because there are only two degrees of freedom in the original model for which the control is designed. As the original model becomes more exact (including more modes), the required gain set increases in complexity but is capable of scaling more eigenvectors. For example, repeating the preceding case with a four-DOF control model, one produces the gain set

$$[K] = \begin{bmatrix} 0 & 0 & 1.1 & -0.4 \\ 0 & 0 & -2.2 & 1.1 \\ -2.2 & 4.3 & 0 & 0 \\ 0.7 & -2.2 & 0 & 0 \end{bmatrix} 742EI \tag{48}$$

which requires four control actuators and sensors to implement. With this new gain set, twice as many eigenvectors are scaled when compared with Eq. (47), producing a response similar to that found in Fig. 6. However, increasing the number of scaled eigenvectors also generates larger control forces, with the maximum control effort generated using Eq. (48) being almost 2.5 times greater than the maximum control effort generated using Eq. (47) for the same random disturbance.

Conclusions

This study presents a feedback control technique known as eigenvector scaling, which may be used to produce mode localization. With this method, the analytic solutions of both the original and mode-localized systems are easily obtainable and may be analyzed to determine the effects of this control approach on the response of the system.

In summary, two observations of particular importance may be made about this eigenvector scaling control approach.

1) Nonzero conditions within the area of localization will have less effect on the rest of the system motion as the degree of localization d increases. If there are no force inputs or other initial disturbances outside the area of localization, then the system will act as an isolator.

2) Nonzero conditions outside the area of localization will increasingly affect the motion within the area of localization as d increases, causing the system to act as a sensor or local amplifier.

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